

Life-Cycle Asset Allocation with Annuity  
Markets: Is Longevity Insurance a Good  
Deal?

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## **Abstract**

We derive the optimal portfolio choice over the life-cycle for households facing labor income, capital market, and mortality risk. In addition to stocks and bonds, households also have access to incomplete annuity markets offering a hedge against mortality risk. We show that a considerable fraction of wealth should be annuitized to skim the return enhancing mortality credit. The remaining liquid wealth (stocks and bonds) is used to hedge labor income risk during work life, to earn the equity premium, and to ensure estate for the heirs. Furthermore, we assess the importance of common explanations for limited participation in annuity markets.

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# Life-Cycle Asset Allocation with Annuity Markets: Is Longevity Insurance a Good Deal?

Studying household financial problems in long-term portfolio choice models is challenging because it requires to consider stochastic investment opportunity sets, illiquid assets such as labor income, housing or deferred tax accounts, and mortality risk (Campbell, 2006). Beginning with Merton (1971) many studies have analyzed the magnitude of hedging demands on the long term asset allocation caused by time-varying investment opportunity sets. This particular string of the life-cycle literature highlights that investors should actively trade stocks, bonds, and money market over time. To this end, interest rate risk has been addressed by Brennan and Xia (2000) and by Wachter (2003), risky inflation by Campbell and Viceira (2001) and Brennan and Xia (2002), and for changing risk premia see Brandt (1999), Campbell and Viceira (1999), Wachter (2002), and Campbell, Chan, and Viceira (2003).<sup>1</sup>

A second string of life-cycle articles emphasize that illiquid assets such as human capital and housing wealth play a dominant role apart from financial wealth in the total asset portfolio of the household. The effect of non-tradable risky human capital (i.e. labor income) on portfolio choice has been addressed by Bodie, Merton, and Samuelson (1992), Heaton and Lucas (1997), Viceira (2001) as well as Cocco, Gomes and Maenhout (2005). As human capital is a closer substitute to bonds than to stocks, young households compensate for the overinvestment in bonds by holding higher stock fractions in financial wealth. Over the life-cycle the optimal stock fraction decreases because the value of human capital declines. The asset allocation problem including housing wealth has been studied by Campbell and Cocco (2003), Cocco (2005), and Yao and Zhang (2005). Furthermore, Heaton and Lucas (2000) consider the importance of entrepreneurial risk, while Faig and Shum (2002) take into account personal illiquid projects such as housing or private business

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<sup>1</sup>The long run implications of stochastic stock market volatility have been addressed by Chacko and Viceira (2006). Also the long term implications of estimation risk about the equity premium have been addressed (see Barberis (2000), and Brennan and Xia (2001)).

in order to explain limited stock market participation.<sup>2</sup> While the list of prior studies related to household financial problems is clearly not exhaustive, the underlying theme of the life-cycle asset allocation literature is that a single feature is highlighted and modeled in order to isolate the relevant economic insights as far as the asset allocation and savings behavior are concerned.

While prior life-cycle asset allocation studies have already included mortality risk by incorporating a stochastic investment horizon, the literature has seldom considered life contingent claims to hedge the mortality risk so far. In addition, the interaction between the life contingent claim and financial wealth (stock and bond investments) has not been fully understood yet. Constant life annuities are the most prominent claim contingent on the individual's survival. A constant life annuity is a financial contract between a buyer (annuitant) and a seller (insurer) that pays out a constant periodic amount for as long as the buyer is alive, in exchange for an initial premium (Brown et al., 2001). In this way, the annuitant transfers the mortality risk to the insurer. The insurer collects the premiums and invests them in riskless bonds in order to meet liabilities arising from guaranteed constant payouts. If the number of annuitants is sufficiently high the independent mortality risks are perfectly hedged through diversification. Surviving annuitants receive the funds of the cohort members who die. This excess return is called the mortality credit and hedges the longevity risk. Annuities are illiquid, as the initially paid premium cannot be recovered anymore by selling the annuity.<sup>3</sup> Thus, the integration of life annuities in a portfolio choice framework requires to deal with sequential real option decisions as life annuity purchases can occur anytime and are irreversible.

In a seminal study, Yaari (1965) finds that all assets should be annuitized because of the mortality credit if the individual is a rational investor without a bequest motive. In his model, the investor is only exposed to mortality risk and all annuities are fairly priced from

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<sup>2</sup>Also, the tax implications on the portfolio of a household have been studied by Dammon, Spatt, and Zhang (2001,2004) and Gomes, Michaelides, and Polkovnichenko (2006).

<sup>3</sup>This is due to a severe problem of asymmetric information about the annuitant's health status between the annuitant and the insurer (or buyer in a secondary market).

an actuarial standpoint. His model abstracts from other sources of risk (e.g. interest rates, stock market, and inflation risk). Recently, Davidoff, Brown, and Diamond (2005) show in a three period model that the conditions under which full annuitization is optimal are not as demanding as the ones set out in Yaari (1965). If there is no bequest motive and the return on the annuity is greater than that of the reference asset, an individual will fully annuitize financial wealth in the presence of a complete market. Partial annuitization may become optimal, if the assumption about complete markets is relaxed. If the investor has a bequest motive, partial annuitization will also be optimal.

We contribute to the prior literature by including constant life annuity markets into a realistically calibrated life-cycle asset allocation framework. This allows us to quantify the hedging demand for mortality risk and to assess the impact on the asset allocation of the financial assets stocks and bonds. Mortality risk can basically have two adverse developments from a life-time consumption and savings perspective. On the one hand, the investor can run out of savings and fall into poverty before dying. The literature refers to this specification as longevity risk. On the other hand, the investor can die early without fully consuming all savings (brevity risk). Although mortality risk clearly influences the optimal savings and consumption policy of the household, mortality risk will have little impact on the composition of the financial wealth portfolio, if only the non-life-contingent assets (stocks and bonds) are considered as (dis-)savings vehicles.

Considering life annuities translates to a portfolio choice problem with illiquid assets. The question arises whether the mortality credit paid by the life annuities is high enough to compensate for the loss of flexibility induced by the irreversibility of the annuity purchase. Therefore, it is of particular interest how the illiquid 'asset' life annuity competes with liquid direct investments in stocks and bonds over the entire life-cycle of the household. To answer this question, we endogenize the decision when to purchase illiquid annuities and to what extent (optimal gradual annuitization).<sup>4</sup> To our best knowledge, there is no research

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<sup>4</sup>See Longstaff (2001) for a detailed study on optimal portfolio choice including illiquid assets in a setting in which the decision about the exposure to illiquid assets is endogenized.

done on the optimal asset allocation and optimal gradual annuitization strategy including work-life and retirement.<sup>5</sup> To fill this gap, we use a realistic discrete-time life-cycle asset allocation and consumption model which incorporates the main three sources of risk of an household: untradable labor income risk during work-life, risky stocks, and stochastic time of death. We assume that the utility function of the household is either of the CRRA or Epstein-Zin type and potentially includes a bequest motive. The model is used to derive the optimal dynamic consumption and asset allocation with stocks, bonds, and illiquid constant life annuities. The consideration of labor income risk is crucial to our analysis as it induces the investor to hold liquid buffer-stock savings in order to hedge short-run adverse developments of labor income. This preference for liquid financial savings competes with the illiquid annuity holdings during work-life. Another trade-off results from the incompleteness in the underlying asset structure of annuities. While illiquid annuities offer the riskless bond return plus the mortality credit, liquid stock investments provide the equity premium in expectation. Shortselling restriction prevent the household from engaging in highly leveraged stock investments although it might be optimal because of the bond-like human capital. Due to the untradable income, irreversibility of annuity purchases, and the shortselling restrictions we have to resort to numerical solution techniques (i.e. backward optimization of the recursive utility function).

We also model frictions in the annuity market, namely administration costs and asymmetric mortality beliefs between the insurer and the annuitant. Market frictions and bequest are the major issues raised in the discussion about the observed weak annuity demand in juxtaposition of its theoretical advantages. One explanation for the weak annuity demand might be high loading factors in quoted annuity rates (Friedman and

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<sup>5</sup>Focusing on the retirement period exclusively, Milevsky and Young (2006) point out that the frequent repurchase of life annuities during retirement is optimal. Their problem is of the continuous time barrier control type and involves the numerical solution of a variational inequality. In contrast, Cairns, Blake, and Dowd (2006a) and Kojen, Nijman, and Werker (2006) consider saving for retirement and determine the optimal asset allocation between bonds and stocks when the retiree is forced to annuitize all his wealth at the beginning of the retirement period. Through this restriction, Cairns, Blake, and Dowd (2006a) and Kojen, Nijman, and Werker (2006) keep their model tractable within the standard HJB approach.

Warshawsky, 1990), although Mitchell et al. (1999), using US data, and Finkelstein and Poterba (2002), using UK data, show that annuities are worth more than commonly perceived. We account for administrative costs by using figures from the findings of Mitchell et al. (1999). A second possibility is the adverse selection process and the resulting asymmetric mortality beliefs between the insurer and the buyer of an annuity. People who are healthier are more inclined to purchase annuities than retirees who are less healthy because they do not expect to live that long (see Brugiavini, 1993). We account for mortality asymmetry by using different forces of mortality for pricing annuities and for evaluating utility. The presence of a bequest motive also reduces the individual's desire to annuitize financial wealth (see Bernheim, 2001).

Our findings shed light on the trade-off between the illiquidity of purchased annuities and their mortality credit during work-life and retirement. We find in a stylized case without administration costs and asymmetric mortality beliefs that it will be optimal to invest partially into annuities from age 20 onwards if financial wealth already exceeds a certain level - even though retirement begins in our model at age 65. Later it is optimal to purchase more annuities until the bond investments are crowded out at age 50 and stocks at age 78. Two effects are the main drivers. On the one hand, the individual's labor income risk resolves over time as the career path materializes. So, the household has a lower preference for buffer-stock savings motive as the household becomes older. On the other hand, the mortality credit rises in age. Finally, it is just a matter of time when the mortality credit is high enough to compensate for the inflexibility related to annuities and for the forfeit equity premium. These results underline that the timing of annuitization is key.

By taking administration costs and asymmetric mortality beliefs into account the optimal timing of the first annuity purchase is postponed to age 59 if financial wealth is high enough. If not, the first annuity purchase will be deferred even further. So, the possibility arises that annuity purchases never occur before the time of death. The consideration of bequest motives reduces the optimal fraction of annuities additionally.

The optimal stock fraction still exhibits the typical life-cycle pattern in spite of the inclusion of annuity markets. However, instead of shifting from stocks to bonds over time the household prefers shifting to annuities. Some bonds are only held if the household wants to ensure estate. Finally, the decreasing stock fraction results because of the decreasing bond-like human capital, the reduced need for liquid buffer-stock savings as well as the increasing mortality credit.

In a robustness analysis we show that cuts in public pensions should be compensated for by additional annuity purchases. The optimal annuity fraction is considerably reduced for the following cases. Obviously, high public pension income crowds out the annuity demand because it is a perfect substitute for annuity payouts. Low relative risk aversion encourages the individual to bet just on stock markets. One might expect that investors with low elasticity of intertemporal substitution use annuities to lock in a certain consumption level. Surprisingly, the opposite is the case. Annuity demand is considerably weaker since the investor is more concerned about short-term consumption smoothing as opposed to hedging long-term longevity risk. Bad health propositions also reduce the annuity demand because the individual considers annuities as too costly. Finally, the gender has no impact on the demand for annuities because annuities are priced by gender specific mortality tables. Only the timing of the annuity purchase differs.

Finally, similar to the literature analyzing utility losses from zero stock market participation (e.g. Cocco, Gomes, and Maenhout, 2005) we conduct a welfare analysis that quantifies the utility losses due to limited participation in annuity markets. We show that the neglect of the mortality credit and thus of the mortality hedge can lead to considerable utility losses being equivalent to a reduction in financial wealth up to about 14 percent of accumulated wealth at age 60.

Explaining why annuities are not more prevalent with a behavioral theory is beyond the scope of our paper. However our model could be used to incorporating first-time participation costs into the model (see e.g. Gomes and Michaelides (2005) for the case of

limited stock market participation) in order to reflect the information cost related to the annuitization decision. From the perspective of the household, annuity investments seem to be an extremely complex financial problem because annuity purchases are irreversible and their returns are contingent on age. Therefore, an investor might not feel qualified enough to participate in annuity markets and shies away from the decision. For a discussion of "investment mistakes" of households we refer the interested reader to Campbell (2006).

The remainder of this paper is organized as follows. In section I we describe the investor's optimization problem and the numerical optimization method. In section II we first discuss the model calibration. Then we present the results for our stylized case as well as the cases with costs and bequest motives. Section III continues with a robustness analysis. In section IV, we conduct a welfare analysis based on computations of certainty equivalents. Section V concludes.

## I. The Model

### A. Preferences

The model is time discrete with  $t \in \{0, \dots, T + 1\}$ , where  $t$  is the adult age of the individual and can be calculated as actual age less 19. The individual lives up to  $T$  years. The individual has a subjective probability  $p_t^s$  to survive from  $t$  until  $t + 1$ . Furthermore, the individual has Epstein-Zin utility defined over a single non-durable consumption good. Let  $C_t$  be the consumption level and  $B_t$  the bequest at time  $t$ . Then Epstein-Zin preferences as in Epstein and Zin (1989) are described by

$$V_t = \left\{ (1 - \beta p_t^s) C_t^{1-1/\psi} + \beta \mathbb{E}_t \left[ p_t^s V_{t+1}^{1-\rho} + (1 - p_t^s) k \frac{(B_{t+1}/k)^{1-\rho}}{1 - \rho} \right]^{\frac{1-1/\psi}{1-\rho}} \right\}^{\frac{1}{1-1/\psi}}, \quad (1)$$

where  $\rho$  is the level of relative risk aversion,  $\psi$  is the elasticity of intertemporal substitution,  $\beta$  is the discount factor and  $k$  the strength of the bequest motive. Since  $p_T^s = 0$  (1) reduces

in  $T$  to

$$V_T = \left\{ C_T^{1-1/\psi} + \beta \mathbb{E}_T \left[ k \frac{(B_{T+1}/k)^{1-\rho}}{1-\rho} \right]^{\frac{1-1/\psi}{1-\rho}} \right\}^{\frac{1}{1-1/\psi}}, \quad (2)$$

which gives us the terminal condition for  $V_T$ .

### *B. Labor Income Process*

Bodie, Merton, and Samuelson (1992), Heaton and Lucas (1997), Viceira (2001) as well as Cocco, Gomes and Maenhout (2005) among others emphasize the importance of incorporating risky labor income into the asset allocation analysis of households. We also include the labor income risk because we consider labor income risk as key when analyzing the trade-off between the inflexibility related to annuity investments and their mortality credit during work-life. Labor income risk creates the demand of the household for liquid assets to hedge adverse developments in labor income (buffer-stock savings). In line with Cocco, Gomes, and Maenhout (2005) the process of labor income follows

$$Y_t = \exp(f(t)P_t)U_t, \quad (3)$$

$$P_t = P_{t-1}N_t. \quad (4)$$

$f(t)$  is a deterministic function of age to recover the hump shape of income stream.  $P_t$  is a permanent component with innovation  $N_t$  and  $U_t$  is a transitory shock. The logarithms of  $N_t$  and  $U_t$  are normally distributed with means zero and with volatilities  $\sigma_N$ ,  $\sigma_U$ , respectively. The shocks are assumed to be uncorrelated. In retirement ( $t > K$ ), we assume for the sake of simplicity that the individual receives constant pension payments  $Y_t = \zeta \exp(f(K))P_K$  after retirement, where  $\zeta$  is the constant replacement ratio. Clearly, it might be worthwhile determining the retirement age  $K$  and labor supply endogenously. This question is beyond the scope of this analysis since we focus on the asset allocation decision.

### C. Incomplete Annuity Market

The individual can invest in an incomplete insurance market by purchasing constant real payout life annuities. A life annuity is a financial contract between an individual and an insurer "that pays out a periodic amount for as long as the annuitant is alive, in exchange for an initial premium" (Brown et al., 2001). The insurance providers themselves can hedge the guaranteed annuity payments by pooling the mortality risks of many annuitants. Contrary to liquid investments, the initial premium cannot be recovered by the individual later on. The actuarial premium  $PR_t$  of a life annuity with payments  $L$  starting in  $t + 1$  is given by:

$$PR_t = La_t, \quad (5)$$

where  $a_t$  is the annuity factor for an individual with adult age  $t$ :

$$a_t = (1 + \delta) \sum_{s=1}^{\infty} \left( \prod_{u=t}^{t+s} p_u^a \right) R_f^{-s}, \quad (6)$$

where  $p_u^a$  are the survival probabilities used by the life annuity provider and  $\delta$  is the expense factor. Thus, the annuity factor is the expense factor times the sum of the discounted expected payouts. Annuities define an asset class with certain age dependent return characteristics because payments are conditional on survival. The funds of those who die in the annuity pool are allocated among the living members of a cohort. This additional return is known as the mortality credit. To get a grasp on the magnitude of the mortality credit, we consider a one-period annuity. The survivor's return from the one-period annuity at age  $t$  is  $R_f/p_t > R_f$ . If the survival probability of the annuitants is 1, the annuitants will - of course - just receive the capital market return  $R_f$ . If  $p_t < 1$ , the return will be larger than  $R_f$ . Obviously, the older the individual, the lower the survival probability  $p_t$ , the higher is the mortality credit. Assume that the real interest rate is  $R_f = 1.02$  and  $p_t = 0.99$  (equivalent to a female survival probability at age 65) then the mortality credit is  $R_f/p_t - R_f = 1.03$  percent. Now assume that  $p_t = 0.95$  (equivalent to an

80 year old female), then the mortality credit is 5.37 percent. In light of realistic equity risk premia ranging between 4 and 6 percent, one can clearly imagine the relevance of the mortality credit.

Our model assumes incomplete annuity markets inasmuch the funds underlying the annuity are invested in bonds only. Thus, the investor can access stock markets via direct investments but not for the portion of assets underlying the annuities. This results in an additional trade-off between the mortality credit and the stock markets equity premium apart from the liquidity trade-off.<sup>6</sup>

#### *D. Capital Market*

The individual can invest via direct investments in the two financial assets: riskless bonds and risky stocks. The real bond gross return denotes  $R_f$ , and the real risky stock return in  $t$  is  $R_t$ . The risky return is log-normally distributed with an expected return  $\mu - 1$  and volatility  $\sigma$ .<sup>7</sup> Let  $\phi_n(\phi_u)$  denote the correlation between the stock returns and the permanent (transitory) income shocks.

#### *E. Mortality*

To give mortality a functional form we apply the Gompertz law for the sake of convenience and because of its widespread use in the insurance and finance literature. Using Gompertz law allows us to model the asymmetry between the insurer's view on mortality and the annuitant's beliefs about the health status in a simple and consistent way. The functional form of the subjective force of mortality  $\lambda^s$  and the force of mortality for computing annuity premiums  $\lambda^a$  are then specified by

$$\lambda_t^i = \frac{1}{b} \exp\left(\frac{t - m^i}{b^i}\right), \quad i = a, s. \quad (7)$$

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<sup>6</sup>The introduction of variable annuities (equity-linked life annuities) to our model is beyond the scope of our analysis. However, constant life annuities play a dominant role in the annuity markets and variable annuities would only further increase the demand for annuities.

<sup>7</sup>We avoid at this point additional state variables in order to keep our problem parsimonious and to account for the curse of dimensionality. However, stock price, labor income, and mortality risks seem to be the dominating risk factors in the household portfolio choice decisions with annuities.

Parameters  $m^i$  and  $b^i$  determine the shape of the force of mortality function. The survival probabilities can now be expressed as follows:

$$p_t^i = \exp\left(-\int_0^1 \lambda_{t+s}^i ds\right) \quad (8)$$

$$= \exp\left[-\exp\left(\frac{t-m^i}{b^i}\right)\left(\exp\left(\frac{1}{b^i}\right)-1\right)\right]. \quad (9)$$

Additionally, we model the subjective force of mortality as linear transformation of the force of mortality derived from the average population mortality table  $\lambda_t^{pop}$  to analyse the impact of bad health propositions. Then we get for the subjective force of mortality and for the subjective probabilities:

$$\lambda_t^s = \nu \lambda_t^{pop}, \quad p_t^s = (p_t^{pop})^\nu. \quad (10)$$

#### *F. Wealth Accumulation*

At each point in time the investor has to make a decision on how to spread wealth on hand  $W_t$  across bonds, stocks, annuities, and consumption. Therefore, the budget constraint is

$$W_t = M_t + S_t + PR_t + C_t, \quad (11)$$

where  $M_t + S_t$  denotes the value of financial wealth,  $M_t$  is the absolute wealth amount invested in bonds and  $S_t$  the amount invested in stocks.  $PR_t$  is the amount that the investor pays for an annuity and  $C_t$  is consumption. The individual's wealth on hand in  $t + 1$  is given by

$$W_{t+1} = M_t R_f + S_t R_{t+1} + L_{t+1} + Y_{t+1}, \quad (12)$$

where  $M_t R_f + S_t R_{t+1}$  denotes the next period value of financial wealth,  $L_{t+1}$  is the sum of annuity payments which the investor gets from previously purchased annuities and  $Y_{t+1}$  is

the labor income. The sum of annuity payments follows the process

$$L_{t+1} = L_t + PR_t/a_t, \tag{13}$$

where  $L_t$  is the sum of all annuity payments from annuities purchased before  $t$  and  $PR_t/a_t$  is the annuity payment purchased in  $t$ . In  $t + 1$  the investor has to make a new decision on how to spread wealth on hand  $W_{t+1}$  across bonds, stocks, annuities, and consumption. At this point, we want to highlight our assumption that the investor can purchase annuities anytime. A large part of the previous literature<sup>8</sup> on dynamic asset allocation with annuities imposed the restriction that annuities can be bought only once and that investors have to completely sell their financial wealth in order to switch to annuities. Yet, stock and bond holdings can be traded dynamically. This asymmetric way of modeling annuities versus stocks and bonds in an optimal dynamic portfolio choice framework lacks consistency. As Milevsky and Young (2006) do, we model annuities just as an additional asset which can be purchased each year. Contrary to Cairns, Blake, and Dowd (2006a) and Kojien, Nijman, and Werker (2006) we prevent households from borrowing against human capital. Otherwise the household would highly leverage stocks. We impose the following restrictions to borrow and to model the irreversibility of annuity purchases:

$$M_t, S_t, PR_t \geq 0. \tag{14}$$

If the individual dies, bequest  $B_t$  will be given by  $B_t = M_{t-1}R_f + S_{t-1}R_t$ . Thus, pension income or annuities cannot contribute to backing bequest motives.

### *G. The Numerical Solution of the Optimization Problem*

The individual's problem is to make a choice each year how much to consume, how much to save in stocks and bonds, and how much to invest in life annuities. Thereby, (1)

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<sup>8</sup>E.g. Blake, Cairns, and Dowd (2003), Cairns, Blake, and Dowd (2006a), Kingston and Thorp (2005), Kojien, Nijman, and Werker (2006), Milevsky and Young (2006), and Milevsky, Moore, and Young (2006).

has to be maximized under budget and short-selling restrictions (11),(12), and (14). The optimal policy depends on four state variables: the permanent income  $P_t$ , wealth on hand  $W_t$ , annuity payouts from previously purchased annuities  $L_t$ , and age  $t$ . As an analytic solution to this problem does not exist, we use dynamic programming techniques to maximize the value function by backward induction.

First of all, the curse of dimensionality (Bellman, 1961) can be partly mitigated by reducing the state space by one state variable. We exploit the scale independence of the optimal policy if we rewrite all variables using lower-case letters as ratios of the permanent income component  $P_t$ . It follows that the indirect utility function can then be rewritten as

$$v_t = \left\{ (1 - \beta p_t^s) c_t^{1-1/\psi} + \beta \mathbb{E}_t \left[ \left( p_t^s v_{t+1}^{1-\rho} + (1 - p_t^s) k \frac{(b_{t+1}/k)^{1-\rho}}{1-\rho} \right) N_{t+1}^{1-\rho} \right]^{\frac{1-1/\psi}{1-\rho}} \right\}^{\frac{1}{1-1/\psi}}, \quad (15)$$

where the only state variables are normalized wealth on hand  $w_t$  and normalized annuity payouts  $l_t$  and age  $t$ . The optimization problem is

$$\max_{\{pr_t, s_t, m_t, c_t\}_{t=0}^{t=T}} v_0, \quad (16)$$

subject to

$$\begin{aligned} w_t &= m_t + s_t + pr_t + c_t && \forall t \\ m_t, s_t, pr_t &\geq 0 && \forall t \\ w_{t+1} &= (m_t R_f + s_t R_{t+1}) N_{t+1}^{-1} + l_{t+1} + \exp(f(t+1)) U_{t+1} && \forall t < K \\ l_{t+1} &= (l_t + pr_t/a_t) N_{t+1}^{-1} && \forall t < K \\ w_{t+1} &= m_t R_f + s_t R_{t+1} + l_{t+1} + \zeta \exp(f(K)) && \forall t \geq K \\ l_{t+1} &= l_t + pr_t/a_t && \forall t \geq K. \end{aligned} \quad (17)$$

We solve the problem in a three-dimensional state space by backward induction. The continuous state variables normalized wealth  $w$  and normalized annuity payouts  $l$  have to

be discretized and the only discrete state variable is age  $t$ . The size of the grid is  $40(w) \times 20(l) \times 81(t)$ . The grid we use is equally spaced for the logarithms of  $w$  and  $l$  since the policy functions are especially sensitive in the area with low  $w$  or  $l$ . For each grid point we calculate the optimal policy and the value of the value function. Thereby, the (multiple) integrals of the expectation term in (15):

$$\begin{aligned} & \iiint \left( p_t^s v_{t+1}^{1-\rho} + (1 - p_t^s) k \frac{(b_{t+1}/k)^{1-\rho}}{1-\rho} \right) N_{t+1}^{1-\rho} \cdot & t < K \\ & \quad \varphi(N_{t+1}, U_{t+1}, R_{t+1}) dN_{t+1} dU_{t+1} dR_{t+1} & (18) \\ & \int \left( p_t^s v_{t+1}^{1-\rho} + (1 - p_t^s) k \frac{(b_{t+1}/k)^{1-\rho}}{1-\rho} \right) N_{t+1}^{1-\rho} \varphi(R_{t+1}) dR_{t+1} & t \geq K, \end{aligned}$$

with  $\varphi(\cdot)$  denoting the (multivariate) probability density function of the log-normal distribution is computed by resorting to gaussian quadrature integration and the optimization is done by numerical constrained maximization routines. We derive the policy functions  $s(w, l, t)$ ,  $m(w, l, t)$ ,  $pr(w, l, t)$ ,  $c(w, l, t)$  and the value function  $v(w, l, t)$  by cubic-splines interpolation.

## II. Optimal Asset Allocation with Annuities

### A. Stylized Case: Fairly Priced Annuities and No Bequest Motive

As a reference case, we will first compute the optimal annuitization and asset allocation strategy for a stylized setting excluding administration costs for annuities, asymmetries in the insurer's and annuitant's mortality beliefs, and bequest motives.<sup>9</sup> This approach allows us to introduce frictions of annuity markets step-by-step in order to show their effects on the optimal policy functions. In a second step, we will introduce administration costs of annuities and for asymmetric mortality beliefs. In a third step, we include bequest motives.

In the stylized case, we assume symmetric beliefs by fitting the subjective and annuitant mortalities to the 2000 Population Basic mortality table for US females.

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<sup>9</sup>This is a common simplifying assumption in prior annuity related studies. See for example Blake, Cairns, and Dowd (2003), Cairns, Blake, and Dowd (2006a), Kingston and Thorp (2005), Koijen, Nijman, and Werker (2006), Milevsky and Young (2006), and Milevsky, Moore, and Young (2006).

Applying nonlinear least squares we fit the Gompertz force of mortality so that the estimated parameters are  $m_f^{(s,a)} = 86.85$ ,  $b_f^{(s,a)} = 9.98$ . We also assume no additional administration charges on the annuity premium ( $\delta = 0$ ) and a zero-bequest motive ( $k = 0$ ). The starting age is set at 20, the retirement age at 65 ( $K = 46$ ), and the maximum age at 100 ( $T = 81$ ). The preference parameters are set to standard values found in the life-cycle literature (e.g. Gomes and Michaelides, 2005): coefficient of relative risk aversion  $\rho = 5$ , elasticity of intertemporal substitution  $\psi = 0.2$ , and discount factor  $\beta = 0.96$ . By setting  $\psi = 1/\rho$  we assume that in the base-line case the investor has CRRA utility. In the robustness section, we will do a sensitivity analysis around the utility parameters of the base-case. Especially, changing the elasticity of intertemporal substitution is interesting since it reflects the investor's willingness to vary consumption levels over time in the context of annuities providing lifelong constant payouts.

The deterministic age-dependent labor income function  $f(t)$  is taken from Cocco, Gomes, and Maenhout (2005). The functional dependence reproduces a hump shaped income profile. Like Gomes and Michaelides (2005) we select volatility parameters for individuals with high school education but without college education and set them to  $\sigma_u = 0.15$  and  $\sigma_n = 0.1$  which is in line with the estimates found by Gourinchas and Parker (2002). The replacement ratio including accumulated pensions from Social Security but excluding voluntary annuitization is set to 68.2 percent as currently estimated by Cocco, Gomes, and Maenhout (2005). Further, we set the real interest rate  $R_f$  to 2 percent, the equity premium  $\mu - R_f$  to 4 percent and stock volatility  $\sigma$  to 18 percent which is in line with the recent life-cycle literature (see for example Cocco, Gomes, and Maenhout (2005), and Gomes and Michaelides (2005)). The correlation between the stock returns and the transitory (permanent) income shocks  $\phi_n$  ( $\phi_u$ ) is zero.

*Figure (1) about here*

The upper, middle, and lower graph of Figure (1) show the optimal holdings in

annuities, stocks, and bonds respectively. The holdings depend on the state variables adult age  $t$  and wealth on hand  $w$  while the level of payouts from previously purchased annuities is set to  $l = 0$ . Thus, Figure (1) depicts the optimal policy space for the case that no annuities have been purchased before. For instance, if the individual is 40 years old and has a wealth on hand of  $w = 500$ , then the optimal strategy will be to invest about  $pr(w = 500, l = 0, age = 40) = 150$  into the annuities, 250 into stock markets, and 40 into bond markets, while the remaining wealth will be spent on consumption.

Even though the recent retirement literature regarding annuities and common wisdom suggest treating payout life annuities as a vehicle to realize consumption after the individual retires, we find the following for the stylized case: The individual would invest into annuities right from the beginning if wealth on hand was above the annuitization (age-wealth) frontier dividing the annuitization and no annuitization region. For instance, in order to start to annuitize at age 20 the individual requires at least a level of wealth on hand of about  $w = 500$ . The required level falls as the individual becomes older, e.g. to  $w = 100$  by age 60.

It becomes apparent that annuities fully crowd out first bonds and then stocks over time, although annuity investments are irreversible whereas bond investments are not. Over the life-cycle the mortality credit increases since the individual profits from outliving other annuitants in the pool. The older the individual gets the more attractive the annuity becomes in the investment opportunity set. The illiquidity drawback of annuities is reduced over the life-cycle as the career path of the household materializes. From age 50 on, the mortality credit is high enough to crowd out bonds totally. After this, the flexibility needed to hedge short-term adverse developments in labor income is fully achieved by liquid stock investments and annuity payouts. It is important to note that annuity payouts are flexible in the way they can be used to protect against income shocks or in the way they can be reinvested into stocks, bonds, and annuities. Therefore, the individual holds stocks in financial wealth and bonds in annuities, only.

The optimal stock demand decreases with financial wealth and age. This result is in line with recent life-cycle literature (e.g. Heaton and Lucas (1997), Viceira (2001) as well as Cocco, Gomes and Maenhout (2005)) and with recommendations made by practitioners as well as policy makers. This optimal stock demand can be explained by considering the individual's entire augmented wealth consisting of financial wealth and human capital, i.e. the present value of future labor income. Even though labor income is risky and uninsurable, the return from human capital (i.e. the labor income) is always positive. Therefore, bonds or life annuities are considered a closer substitute for human capital than stocks. In turn, the decrease in human capital is compensated for by holding life annuities and bonds. The optimal stock holdings decrease to 0 by age 78. Three factors are relevant for the sinking stock market participation over the life-cycle: First, the decreasing human capital, second, the increasing mortality credit, and third, the decreasing need for flexibility. From this age 78 on, the individual holds no liquid wealth but annuities instead. Now full annuitization is optimal because the mortality credit offsets the opportunity costs arising from illiquidity and the foregone equity premium. Finally, at the end of the life-cycle (age > 78) the optimal annuity holdings decrease because the individual's time preference for consumption rises.

The upper graph of Figure (3) depicts the expected life-cycle profile for savings, labor income, consumption, and annuity purchases. We compute the expectations by resorting to Monte-Carlo simulations. Thereby, we assume that the individual has no initial endowment apart from the labor income. Strikingly, the amount of liquid savings the individual accumulates over work life is rather low compared to the previous results found in the life-cycle literature. At age 52, savings just peak at about 5 times the labor income. Gradually, savings decline to 0 by age 78. For the remaining lifetime, the individual neither faces labor income risk nor stock market risk. The only uncertainty at this point is the mortality risk.

The upper graph of Figure (4) exhibits the expected asset allocation. It becomes

apparent that the investor gradually switches from stocks into annuities. The individual starts to buy annuities from age 40 onwards and continues to purchase annuities until age 99. However, from age 78 on, the household only uses small fractions of pension and annuity income to purchase annuities. The household consumes less than the labor income until 48 in order to build up financial and annuity wealth. Table I shows that expected consumption increases until the end of the life-cycle because the ongoing annuity purchases lead to higher payouts, e.g. at age 75, expected payouts are already 3.3 percent higher than the regular pension income.

### *B. Implications of Administration Costs and Asymmetric Mortality Beliefs*

This section introduces frictions to the annuity market by adding explicit costs in terms of administration expenses and implicit costs in terms of asymmetric mortality beliefs. The expense factor  $\delta$  is set to 7.3 percent to reflect the explicit costs. This factor is taken from the 1995 annuity value per premium dollar computed on an after tax basis by Mitchell et al. (1999). This factor can also be seen as a risk premium that is necessary to hedge mortality risk by organizing an insurance pool. The risk premium is also needed to compensate the insurance provider for the residual risk realized when the number of annuitants inside the pool is finite and when mortality can evolve stochastically over time.<sup>10</sup> In order to price the annuity we use the 1996 US Annuity 2000 Basic mortality table while we still use 2000 Population mortality table for the subjective survival probabilities. The fitted parameters are  $m_f^a = 90.51$ ,  $b_f^a = 8.73$  and  $m_f^s = 86.85$ ,  $b_f^s = 9.98$ , respectively. The insurance provider calculates annuity premiums by using higher survival probabilities as a result of the adverse selection process. In this process, individuals who believe themselves to be healthier than average are more likely to buy annuities (e.g. Brugiavini (1993)). The higher the asymmetry in mortality beliefs, the higher are the implicit costs of annuities.

*Figure (2) about here*

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<sup>10</sup>We refer the interested reader to the article Cairns, Blake, and Dowd (2006b) for a further discussion of the implications of stochastic mortality on the pricing of mortality contingent claims.

The introduction of both explicit and implicit costs reduces the return on the annuity dramatically. A numerical example should illustrate the effect. While in the no cost scenario a 65-year-old woman receives a yearly payout of USD 6,474 in return for a USD 100,000 annuity premium, in the alternative case with costs, the payout decreases to USD 5,360. The explicit and implicit costs lead to an effective reduction in payouts of 20.79 percent in this example.

Figure 2 left panel shows the additional costs postpone the optimal timing of annuity purchases and reduce the overall level of annuity holdings. Whereas in the stylized case the individual partially annuitizes from age 20 on for  $w > 500$ , the introduced costs lead to a deferral of the first annuity purchase to age 59, 6 years prior to retirement. Until this age, the individual holds riskless bonds instead of annuities to control for the risk exposure of the financial wealth. From age 59 on, the individual gradually shifts from bonds into annuities. In turn, the optimal direct bond investment is 0 from age 65 on. The mortality credit is sufficiently high to compensate for the irreversibility related to annuity purchases compared to bond holdings. Since the mortality credit is reduced by the explicit and implicit costs, it is optimal to postpone full annuitization from age 78 in the no-cost scenario to age 82 with costs. Interestingly, it might be the case that the individual will never buy annuities at all if financial wealth stays below the participation frontier. In the lower area, the individual prefers to hold stocks in order to compensate for the over-investment in bond-like human capital.

*Table I about here*

The middle graph of Figure (3) shows the expected life-cycle profile for the case including costs. Obviously, the consideration of costs has a significant impact on the asset location decision of the household. Optimal expected financial wealth is twice as high as in the stylized case because annuity purchases are postponed by 19 years to age 59 compared to the previous case. From age 59 on, the individual suddenly shifts considerable amounts

of liquid financial wealth to annuities. While in the previous case slowly reducing liquidity needs cause the gradual increase of annuity purchases during work-life, return considerations dominate the trade-off between financial wealth and annuities in the case including costs. The middle graph of Figure (4) offers the reason for the jump in annuity purchases. Annuity purchases are financed by selling bonds at age 59 because from that age on the riskless return of annuities after costs is higher than the one of bonds. Table I reveals that the payouts are still high enough to cushion the drop in income of 31.8 percent at the beginning of the retirement period although payouts drop due to costs from 63.9 to 41.2 percent of pension income ( $L/Y$ ) at age 65.

### *C. Implications of Bequest Motives*

Empirical studies such as Kotlikoff and Summers (1981) find that almost 80 percent of the total accumulated wealth in the United States is due to intergenerational transfers. This fact raises the question as to whether bequests are accidental or intentional. The literature on intentional bequests distinguishes between altruistic and strategic bequest motives as opposite ends of the spectrum. For instance, Abel and Warshawsky (1988) study the altruistic bequest motive in a reduced form and find a joy of giving parameter that is of a substantial magnitude. Bernheim, Shleifer, and Summers (1985) analyze the strategic bequest motive and discover empirical evidence. By contrast, Hurd (1987) does not find any evidence of bequest motives because the pattern of asset decumulation is similar among different household sizes. In addition, Hurd (1989) can support his prior findings by showing that the nature of most bequests is accidental because the date of death is uncertain to an individual.

*Figures (3) and (4) about here*

The strength of the bequest motive controls the individual's preference for liquid savings. We set the bequest strength parameter to  $k = 2$ . Because of the positive

probability to die, the individual holds some liquid wealth throughout the life-cycle to ensure estate. Thus, holding liquid wealth is optimal for three reasons in our model: first, in order to cushion adverse income shocks during work-life, second, in order to participate in the equity premium, and third, in order to back liabilities arising from bequest motives. The right panel of Figure (2) reveals that the introduction of the bequest motive postpones the time of the first annuity purchase because a higher mortality credit is required as a compensation for the lack of bequest potential related to annuities. Consequently, the annuitization frontier shifts to higher wealth levels. Then, it becomes more likely that the individual will never annuitize if financial wealth remains at low levels over the entire life-cycle.

Again, the stock demand monotonically decreases in age and grows with wealth. Stock holdings never fall to 0 in order to meet the individual's risk appetite for the estate. The optimal bond demand is crowded out by annuities but increases again at very high ages ( $> 80$ ) in order to back the bequest motive.

The expected life-cycle in the lower graph of Figure (3) shows that liquid savings peak at age 59 about 10 percent higher compared to the case without a bequest motive but with costs. Afterwards, savings decline more slowly because the individual requires liquidity to bequeath the heirs. Even at age 100 the estate is about 4 times higher than the annual pension income in expectation. Although this indicates a rather strong bequest motive the household still buys considerable amounts of annuities. Consequently, Table I shows that at age 65 the expected payout remains at 41.2 percent of the pension income identical to the case without bequest. However, later on in life, the payouts are below those in the case without a bequest motive as only little further annuity purchases occur. The bottom graph of Figure (4) indicates that direct bond investments make up a higher fraction of liquid and annuity wealth than in the cases before. The bond fraction becomes dominant for very old individuals as the value of annuities decreases with age.

### III. Robustness Analysis

#### A. Implications of Gender Specific Mortalities and Health Status

In our robustness analysis we vary parameters based on the case with costs and bequest we have discussed before. Since men show statistically lower life-expectancies than women (e.g. 74.8 years versus 80.1 years in the United States), this section determines the effects on the men's optimal annuity holdings. In order to account for bad health propositions this section also considers individuals who have a lower than average life-expectancy.<sup>11</sup> Higher asymmetries in mortality beliefs result in higher implicit costs making the return characteristics of annuities less attractive compared to stocks and bonds. To double the force of mortality, we set the parameter  $\nu = 2$ .<sup>12</sup>

*Table II about here*

The expected optimal annuity holdings for men are reported in Table II. We assume that the male annuitant has to pay administration expenses ( $\delta = 0.073$ ). He also has a bequest motive ( $k = 2$ ). To account for the lower life-expectancy of men we use male mortality tables. In turn, both the subjective and the annuitant survival probabilities decrease. Table II clearly shows that men also seek a mortality hedge although their life-expectancy is much lower than the women's. However, there is one important difference in the policy between the genders. While women start to annuitize at age 60 (Figure (3)), men prefer to annuitize earlier at age 56. The reason is that the mortality credit for men increases faster in age than in the case of women because men's survival probabilities are lower. More male annuitants die and pass their funds on to the surviving members of the cohort. For instance, while a female annually receives USD 5,360 in return for a premium of USD 100,000 at age 65, a male gets USD 6,423.

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<sup>11</sup>This has similar effects in expectation as modeling negative health shocks stochastically but is more parsimonious because we can avoid one more state variable.

<sup>12</sup>For the sake of simplicity we omit to model risky health states between the states dead and alive. However, by doubling the force of mortality we can assess the importance of the health status on the asset allocation in a comparative static analysis.

In the case with bad health propositions the asymmetric beliefs in mortality are stronger than in other cases. Therefore, implicit costs of annuities increase and the demand for annuities decreases. The individual has a stronger time preference because the mortality is higher. The motive to postpone consumption to future periods is weaker. The individual either keeps liquid savings or purchases annuities. The mortality credit should be higher in order to compensate for the stronger time preference. Yet, since annuity mortality tables are not adjusted for the health status, the mortality credit remains the same. However, Table II verifies that even the individual with the doubled force of mortality buys some annuities when living longer than expected.

### *B. Relative Risk Aversion and Elasticity of Intertemporal Substitution*

Decreasing the level of relative risk aversion to  $\rho = 2$  and keeping the elasticity of intertemporal substitution constant at  $\psi = 0.2$ , we find that the annuity demand plummets to 0 in expectation (according to Table II) because the mortality credit is too low compared to the equity premium. The individual invests 100 percent of the savings into stocks for a large part of the lifetime in order to gain the equity premium. Only at the very end of the life-cycle some fraction of the financial wealth is held in bonds due to the reduction in the bond-like human capital.

The elasticity of intertemporal substitution controls the individual's willingness to substitute consumption over time. Variations in  $\psi$  should definitely have some impact on the optimal annuity demand since the annuity pays out a constant cash flow stream allowing the individual to hold consumption constant over time. On the contrary, liquid savings in stocks and bonds cannot hedge mortality risk. The individual may run out of funds before perishing. Therefore, we compute two additional cases with  $\psi = 0.1$  and  $\psi = 0.3$ , respectively. We find that changing  $\psi$  has no influence on the optimal stock holdings  $s(w, l, t)$  at all. So the policy remains exactly the same as in the case with bequest and costs in Figure (2). The bond and annuity holdings  $m(w, l, t)$ ,  $pr(w, l, t)$  do not change until age 60. Afterwards one can observe the following: the higher  $\psi$  the more bonds are

substituted by annuity holdings (see also Table II). The higher the elasticity of intertemporal substitution the less is the investor concerned with low-frequency consumption smoothing. Instead, it is optimal to hold annuities in order to gain the mortality credit of annuities in the long run.

### *C. Different Levels of Pension Income*

Nowadays, the pay-as-you-go public pension systems are running into trouble since longer life-expectancies and lower birth rates lead to a decreasing ratio of contributors to beneficiaries. We decrease public pension payments from 68.2 percent to 50 percent of the labor income ( $\zeta = 0.5$ ) at age 65 while keeping the labor income process the same during work life. This implies constant Social Security taxes in the presence of pension cuts.

Table II shows the "crowding-in" effect into the annuity markets. As the pension income is constant over time life annuities are a perfect substitute for pension income. So, the expected optimal annuity fraction rises if public pensions get cut. This result has important implications for pension systems shifting from pay-as-you-go to privately funded systems. Our results clearly show that life annuities have to be considered in the discussion about regulations and policies with respect to the optimal asset allocation of households in the transition process from a pay-as-you-go to a privately funded pension system.

Table II also reports a case in which the replacement ratio is set to 100 percent ( $\zeta = 1$ ). For instance, this may correspond to households having generous defined benefit pensions which also substitute annuity holdings. Since these defined benefit pensions already resemble life annuities the optimal annuity wealth fraction decreases. In expectation, the demand for annuities at age 60 falls from 17.7 percent in the case  $\zeta = 0.682$  to 3.2 percent in the case  $\zeta = 1$ . Only very late in life it is optimal to buy more annuities to skim the risen mortality credit.

## IV. Welfare Analysis

In this section, we conduct a welfare analysis which quantifies the utility losses resulting from limited participation in annuity markets. This welfare analysis is similar to the one in Cocco, Gomes, and Maenhout (2005) in which the authors compute utility losses generated by limited participation in equity markets. The substantial optimal fraction of annuities inside the households portfolio in most settings suggests that considerable utility gains can be reaped from optimally investing in annuity markets in expectation. Surprisingly, we empirically observe a weak participation in annuity market in juxtaposition to the theoretical advantages of annuity purchases. Naturally, theories from behavioral finance might explain the annuity puzzle. One behavioral explanation might be that a household may not feel qualified enough to participate in annuity markets and shies away from real option decisions in order to avoid severe investment mistakes (see also Campbell (2006)). Another behavioral explanation could be that information costs related to annuity markets are considerably high from the perspective of the household. This issue has been addressed by Gomes and Michaelides (2005) among others in the context of the limited equity participation puzzle. However, modeling behavioral motives is beyond the scope of our analysis as we study the asset allocation problem with annuities. Hence, one part of the utility losses derived in our welfare analysis could be generated by behavioral problems.

*Table III about here*

For all cases considered so far, we first compute the expected utility of households living in a world with access to annuity markets. Then, we compute the expected utility of households having no access. Apparently, the expected utility will be always higher for individuals, since annuities expand the decision set. We numerically equate the expected utility of both cases for the age 60, 70, 80, and 90 respectively by raising the financial wealth of households having no access to annuities. The difference is called the equivalent increase in financial wealth required to compensate the household for the lack of annuity markets.

Table III shows that annuity markets imply a considerable rise in financial wealth for the cases in which we derived high annuity fractions. In our stylized case, financial wealth equivalently increases from 14.41 percent at age 60 up to 49.83 percent at age 90. Adding administrative costs and bequest subsequently, we observe an apparent decline of the equivalent increase in financial wealth to 8.01 percent and to 5.69 percent at age 60. The discrepancy becomes considerably smaller for a higher age. The subsequent cases show an equivalent increase in financial wealth of 30.12 percent and 30.07 percent at age 90, respectively.

Men will exhibit particularly high utility gains if they live longer than expected by the annuity mortality table. Bad health propositions imply smaller utility gains reflecting the implicit costs of adverse mortality beliefs. The utility gain remains at low levels even for old individuals since they have purchased less annuities before. The higher the intertemporal elasticity of substitution, the higher is the utility gain from annuities because - as pointed out before - the individual is less concerned with short-term consumption smoothing and is more willing to bet on the mortality credit in the long-run. As expected, the lower the payments from the public pension systems and/or defined benefit plans, the larger are the utility gains as well as the demand for annuities.

*Figure (5) about here*

The equivalent increase in financial wealth can be attributed to the mortality credit financing extra consumption. To demonstrate the advantage in consumption possibilities we conduct a Monte Carlo analysis when computing the 10th, 50th, and 90th percentile of consumption for the cases including and excluding annuity markets. Figure (5) demonstrates that the distribution of consumption (10th, 50th, and 90th percentile) of an household with no annuity holdings is humped-shaped. In the case including annuity markets, the purchase of annuities avoids a shortfall in consumption during retirement.

## V. Conclusion

We introduce incomplete annuity markets into a realistically calibrated life-cycle asset allocation model and derive the optimal dynamic annuitization and asset allocation strategy during work-life and retirement. The integration of life annuities in a portfolio choice framework requires to deal with sequential real option decisions as life annuity purchases are irreversible and can happen anytime. The real option decision is based on an evaluation of the utility trade-off between the return enhancing mortality credit offered by annuities guaranteeing a constant lifelong payout and the opportunity costs related to life annuity purchases. The opportunity costs arise for the following reasons: first, the household loses financial flexibility to react to changes in the state variables: permanent labor income (stochastic during work life), wealth on hand, and age. Second, the household foregoes the equity premium. Third, the household forfeits estate for the heirs as annuity payouts are survival contingent.

We derive the optimal annuitization region in the state space in which the utility gains derived from the mortality credit offset the opportunity costs. The timing of the annuity purchases is crucial. The drivers over time are: increasing mortality credit, sinking value of human capital, and decreasing uncertainty about labor income. In turn, the optimal annuity fraction increases with age. Also, wealth on hand has to exceed a certain age dependent level to satisfy the investor's liquidity preference before it is optimal to purchase annuities. We also assess the impact of administration costs of annuities and implicit costs generated by using higher survival probabilities for computing the annuity premium as compared to those assumed by the household. Eating up the mortality credit, the above costs postpone annuitization and reduce the optimal annuitization fraction substantially. If we exclude bequest motives and the household is old enough, the mortality credit will be sufficiently high to warrant a complete shift from financial wealth to annuity holdings. But, in the presence of bequest motives retaining some financial wealth in terms of stocks and bonds is optimal to ensure estate.

As far as the optimal stock fraction is concerned we still find the typical life-cycle pattern derived in prior life-cycle studies comprising human capital. However, the household would not shift gradually from stocks to bonds but to annuities instead. Thus, the decreasing stock fraction is now a result of the decreasing human capital, the reduced need for liquidity, and the increasing mortality credit. While in the no-bequest case zero participation in stock (and bond) markets becomes optimal late in life, the optimal stock fraction drops in favor of bonds in the case including bequest motives.

Our robustness analysis reveals that the annuity fraction increases with: better health, higher risk aversion and elasticity of intertemporal substitution, and public/occupational pension cuts. The men's optimal annuitization pattern equals the women's, except that men annuitize earlier. We show that administration costs, asymmetric mortality beliefs between the household and insurer, public pensions, and bequest motives clearly reduce the optimal annuity fraction, but cannot explain limited annuitization. Limited annuitization is predicted in cases with bad health, low risk aversion, and already high pre-existing pensions which are perfect substitutes for annuities.

We confirm in a final welfare analysis, which is comparable to the one in Cocco, Gomes, and Maenhout (2005) for the case of limited stock market participation, that also for the case of non-participation in annuity markets utility losses are substantial (up to 14 percent of accumulated wealth) even after considering non-behavioral factors regarded as prominent explanations for limited annuitization.

Our analysis demonstrates that frictions and the bequest motive are not sufficient to explain the limited participation in annuity markets. Our model could be used to include explanations from behavioral finance to explain the annuity puzzle. From the perspective of the household, annuity investments seem to be an extremely complex financial problem because annuity purchases are irreversible and their returns are contingent on survival. Therefore, an investor might not feel qualified enough to participate in annuity markets and

shies away from decisions.<sup>13</sup> This could be done by incorporating first-time participation costs to reflect information costs (see e.g. Gomes and Michaelides (2005) for the case of limited stock market participation).

Finally, in light of the welfare gains for households we deem our study to be extremely relevant for policy makers and regulators defining asset allocation rules for privately funded pension systems. Further, it also shows implications for the optimal design of accumulation and decumulation plans provided by the financial services industry and corporate pension plans.

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<sup>13</sup>For a detailed discussion of "investment mistakes" and financial literacy of households we refer the interested reader to Campbell (2006), and Lusardi and Mitchell (2006).

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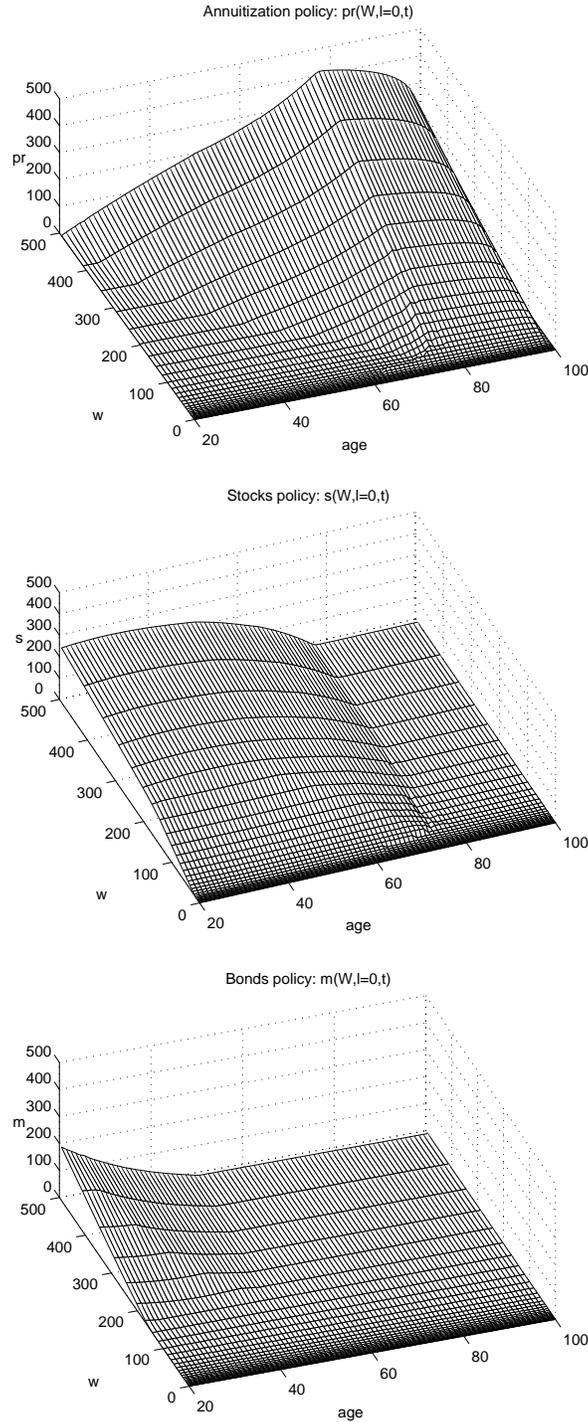
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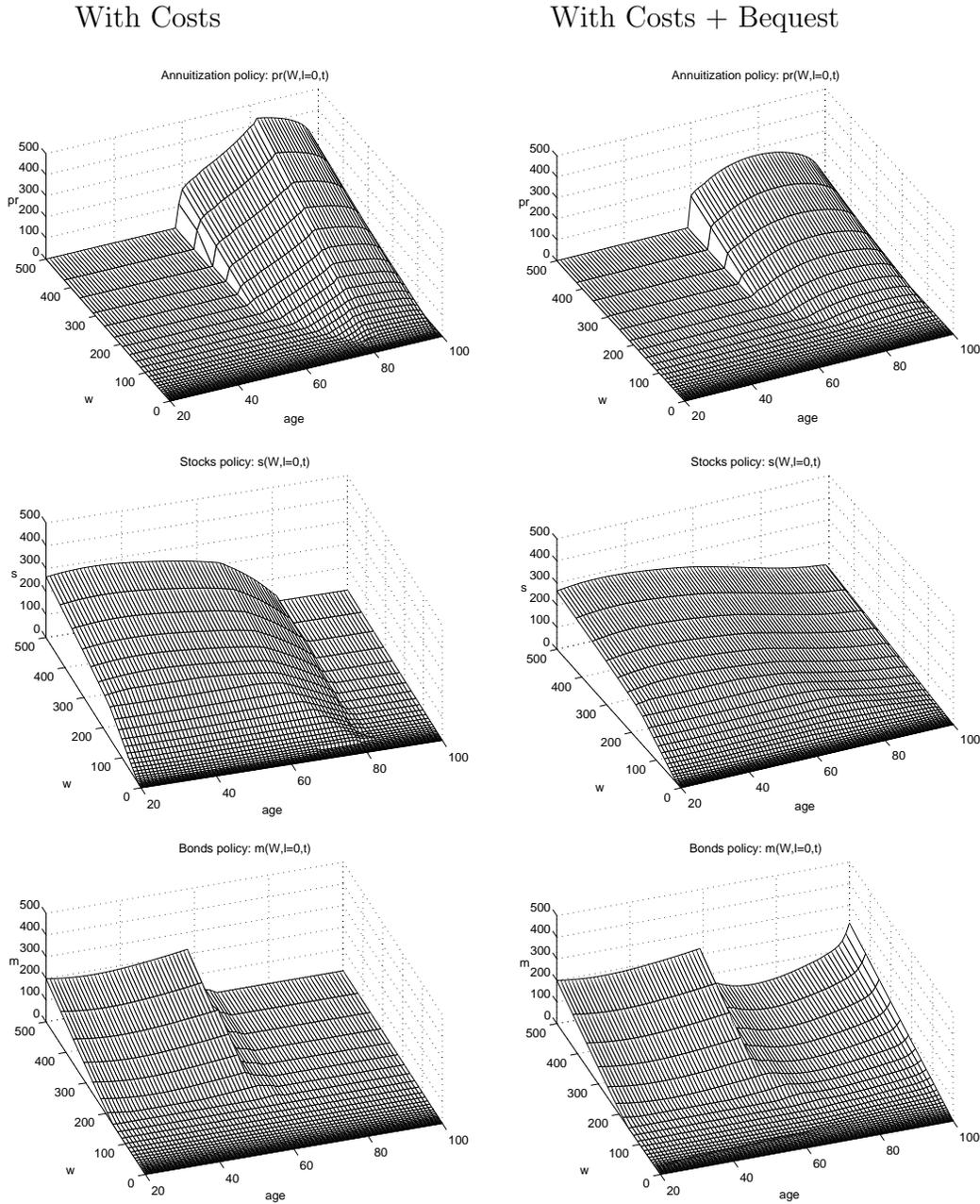
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Figure 1



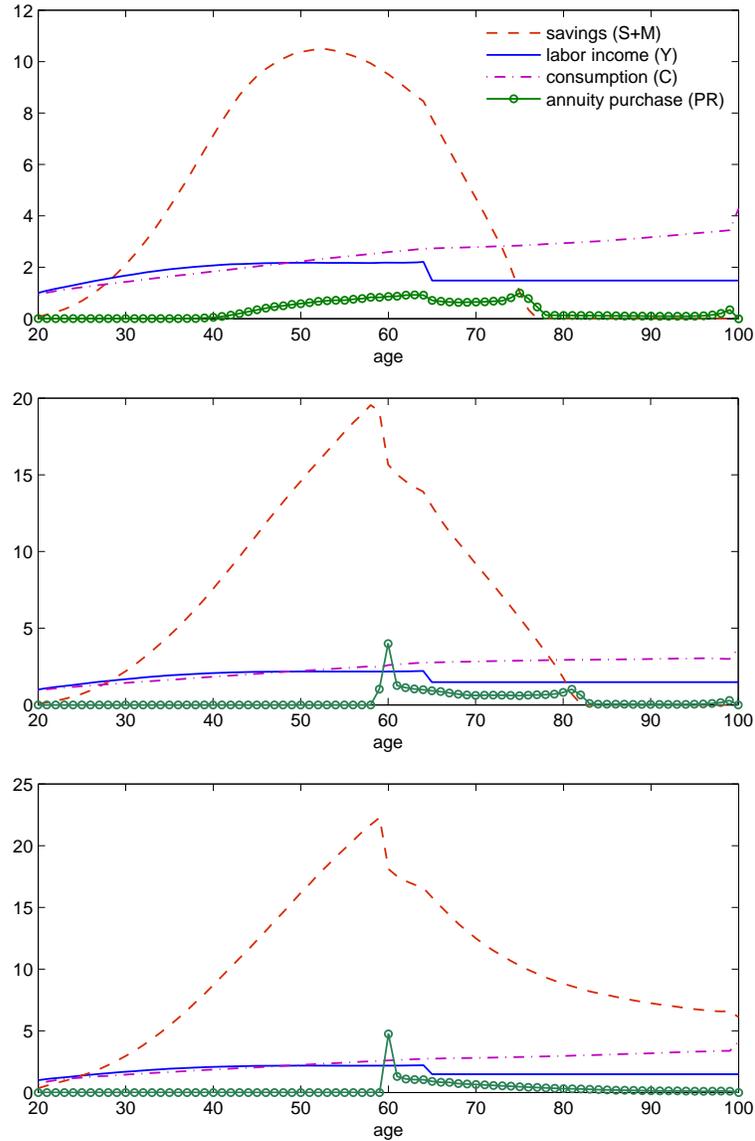
**Figure 1: Optimal Asset Allocation: Stylized Case.** For this case, we assume a female with maximum life-span age 20 - 100, no initial endowment, no administration costs for annuities, no mortality asymmetries,  $RRA = 5$ ,  $EIS = 1/5$ , and a zero-bequest motive ( $k = 0$ ). The figure depicts the optimal holdings of annuities  $pr(w, l, t)$  (upper graph), stocks  $s(w, l, t)$  (middle graph), and bonds  $m(w, l, t)$  (lower graph) depending on the age and wealth on hand  $w$ . The state variable level of annuity payouts is set to  $l = 0$ .

Figure 2



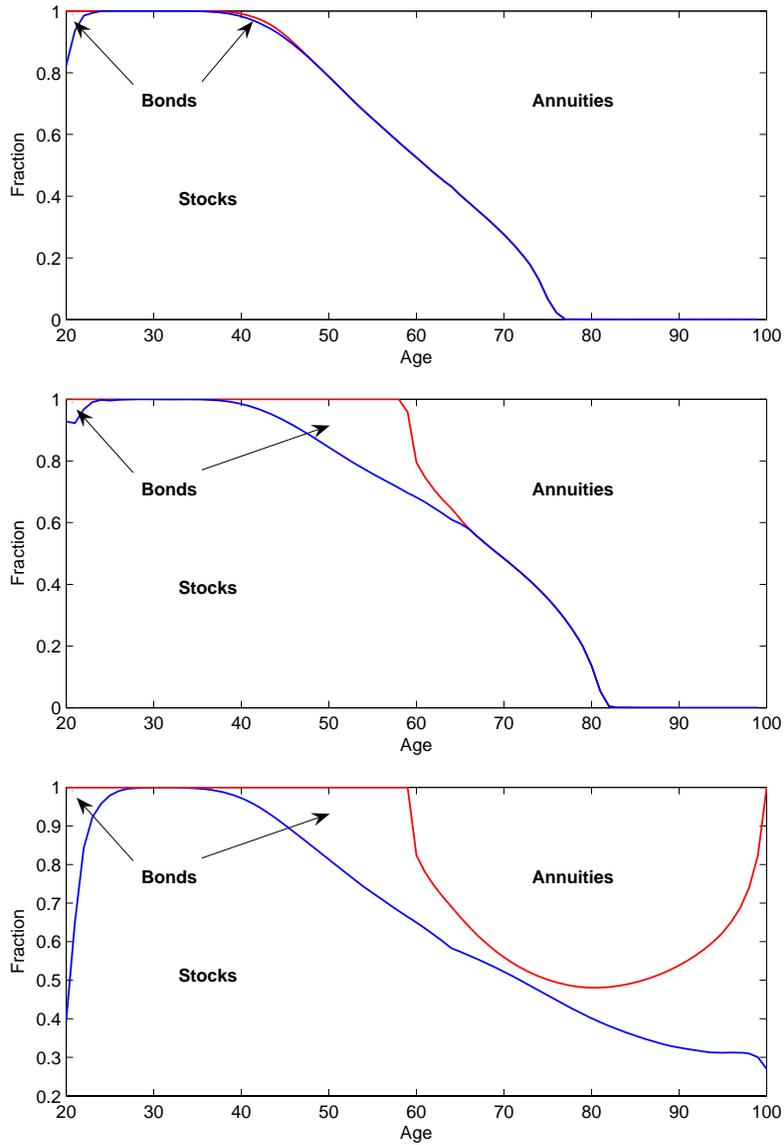
**Figure 2: Optimal Asset Allocation with Costs (Left Column) and Optimal Asset Allocation with Costs and a Bequest Motive (Right Column).** The figure depicts the optimal holdings of annuities  $pr(w, l, t)$  (upper panel), stocks  $s(w, l, t)$  (middle panel), and bonds  $m(w, l, t)$  (lower panel) depending on the age and wealth on hand  $w$ . Left column: for this case, we assume administration costs for annuities  $\delta = 0.073$ , mortality asymmetries (2000 Population Basic vs. 1996 US Annuity 2000 Basic mortality table), and a zero-bequest motive ( $k = 0$ ). Right column: for this case, we additionally assume a bequest motive of ( $k = 2$ ). The state variable level of annuity payouts is set to  $l = 0$ . As in the stylized case, this optimization assumes a female with maximum life-span age 20 - 100, no initial endowment, and  $RRA = 5$ ,  $EIS = 1/5$ .

Figure 3



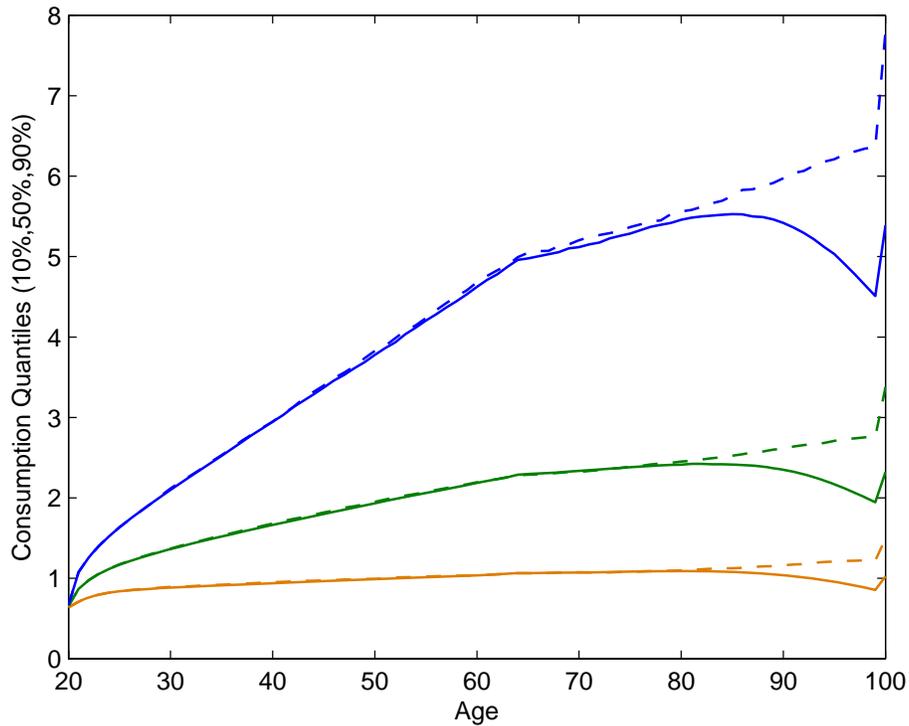
**Figure 3: Life-Cycle Profiles: Stylized Case (Top), with Costs (Middle), and with Costs and Bequest (Bottom).** All three graphs depict expected consumption, labor income, savings, and annuity purchases over the life-cycle. Stylized case: for this case, we assume a female with maximum life-span age 20 - 100, no initial endowment, no administration costs for annuities, no mortality asymmetries,  $RRA = 5$ ,  $EIS = 1/5$ , and a zero-bequest motive ( $k = 0$ ). Case with costs: here, we additionally consider administration costs  $\delta = 0.073$  for annuities and mortality asymmetries (2000 Population Basic vs. 1996 US Annuity 2000 Basic mortality table). Case with costs and bequest: for this case, we add a bequest motive of ( $k = 2$ ) to the case with costs. The expected values are computed by simulating 100,000 life-cycle paths based on the optimal policies derived by the numerical optimization.

Figure 4



**Figure 4: Expected Asset Allocation: Stylized Case (Top), with Costs (Middle), and with Costs and Bequest (Bottom).** All three graphs depict the expected stock, bond, and annuity holdings. Stylized case: for this case, we assume a female with maximum life-span age 20 - 100, no initial endowment, no administration costs for annuities, no mortality asymmetries,  $RRA = 5$ ,  $EIS = 1/5$ , and a zero-bequest motive ( $k = 0$ ). Case with costs: here, we additionally consider administration costs  $\delta = 0.073$  for annuities and mortality asymmetries (2000 Population Basic vs. 1996 US Annuity 2000 Basic mortality table). Case with costs and bequest: for this case, we add a bequest motive of ( $k = 2$ ) to the case with costs. The expected values are computed by simulating 100,000 life-cycle paths based on the optimal policies derived by the numerical optimization.

Figure 5



**Figure 5: Consumption Percentiles (10th,50th, and 90th) with and without Annuities.** The dashed (solid) lines reflect the case with (without) annuities. The blue, green, and orange lines reflect the 90th, 50th, and 10th percentile respectively. The calculations are based on 100,000 Monte-Carlo simulations. We assume the case with bequest and costs.

**Table I**  
**Expected Optimal Annuitization Strategy**

Table I reports the expected optimal annuitization policy for three alternative cases. The first, stylized case assumes a female with maximum life-span age 20 - 100, no initial endowment, no administration costs for annuities, no mortality asymmetries,  $RRA = 5$ ,  $EIS = 1/5$ , and a zero-bequest motive ( $k = 0$ ). The second case introduces administration costs for annuities  $\delta = 0.073$  and mortality asymmetries (2000 Population Basic vs. 1996 US Annuity 2000 Basic mortality table). The third case additionally considers a bequest motive ( $k = 2$ ).  $PR/W$  is the fraction of financial wealth used to purchase new annuities at a certain age.  $L/Y$  reports the magnitude of annuity payouts  $L$  relative to the labor income  $Y$ .  $L/(C - Y)$  depicts how much of the consumption  $C$  not taken from the labor income is backed by annuity payouts. We compute the expected values based on 100,000 Monte-Carlo simulations by using the optimal policy functions  $s(w, l, t)$  (stocks),  $m(w, l, t)$  (bonds),  $pr(w, l, t)$  (annuities), and  $c(w, l, t)$  (consumption). All figures are reported as percentages.

Age	Stylized case			Expected Optimal Annuitization			With costs and bequest		
	$PR/W$	$L/Y$	$L/(C - Y)$	$PR/W$	$L/Y$	$L/(C - Y)$	$PR/W$	$L/Y$	$L/(C - Y)$
25	0	0	0	0	0	0	0	0	0
30	0	0	0	0	0	0	0	0	0
35	0	0	0	0	0	0	0	0	0
40	0.4	0.1	0.1	0	0	0	0	0	0
45	2.3	2.0	12.9	0	0	0	0	0	0
50	3.7	7.8	40.4	0	0	0	0	0	0
55	4.5	16.7	75.1	0	0	0	0	0	0
60	5.6	28.7	116.9	15.4	2.9	38.1	16.5	0	0
65	5.1	63.9	59.9	4.5	41.2	33.8	3.7	41.2	33.4
70	6.2	81.2	75.9	3.8	58.9	48.8	3.2	58.6	48.6
75	15.9	103.3	96.7	5.2	75.1	62.5	2.6	74.1	61.9
80	2.7	123.1	109.2	11.7	96.8	82.1	2.0	87.8	72.2
85	2.4	129.9	107.8	1.0	118.3	103.2	1.4	100.3	77.9
90	2.0	138.3	106.2	0.8	120.6	102.9	1.0	111.9	81.8
95	1.9	149.5	105.6	0.9	124.0	103.1	0.8	123.8	85.3

**Table II**  
**Expected Optimal Asset Allocation for Alternative Cases**

Table II reports the expected asset allocation of the individual's portfolio consisting of stocks, bonds, and annuities. The expected asset allocations are displayed for the age 30, 45, 60, 75. The first, stylized case assumes a female with maximum life-span age 20 - 100, no initial endowment, no administration costs for annuities, no mortality asymmetries,  $RRA = 5$ ,  $EIS = 1/5$ , and a zero-bequest motive ( $k = 0$ ). The second case introduces administration costs for annuities  $\delta = 0.073$  and mortality asymmetries (2000 Population Basic vs. 1996 US Annuity 2000 Basic mortality table). The third case additionally considers a bequest motive ( $k = 2$ ). The remaining cases are variations of the third case with costs and bequest. The value of all annuities the investor holds at a certain age is calculated as the premium the investor would have to pay in order to achieve the same payout stream as with the previously purchased annuities. We compute the expected values based on 100,000 Monte-Carlo simulations by using the optimal policy functions  $s(w, l, t)$  (stocks),  $m(w, l, t)$  (bonds),  $pr(w, l, t)$  (annuities), and  $c(w, l, t)$  (consumption). All figures are reported as percentages.

Case	Stock fraction (%)				Bond fraction (%)				Annuity fraction (%)			
	30	45	60	75	30	45	60	75	30	45	60	75
Stylized case	100.0	91.3	52.4	6.7	0.0	0.9	0.0	0.0	0.0	7.8	47.6	93.3
With costs	100.0	93.0	68.6	35.8	0.0	7.0	11.3	0.0	0.0	0.0	20.1	64.2
With bequest	100.0	90.4	64.8	45.9	0.0	9.6	17.4	4.1	0.0	0.0	17.7	50.0
Males	99.9	89.6	59.8	40.9	0.1	10.4	1.9	7.7	0.0	0.1	38.3	51.5
Bad health	99.9	90.3	65.2	48.6	0.1	9.7	26.6	18.5	0.0	0.0	8.2	32.9
Low IES ( $\psi = 0.1$ )	99.6	92.2	67.8	49.8	0.4	7.8	23.2	12.6	0.0	0.0	9.0	37.5
High IES ( $\psi = 0.3$ )	100.0	88.3	61.6	41.3	0.0	11.7	12.2	0.0	0.0	0.0	26.2	58.7
Low RRA ( $\rho = 2$ )	100.0	100.0	100.0	97.9	0.0	0.0	0.0	2.1	0.0	0.0	0.0	0.0
Low pension income ( $\lambda = 0.5$ )	100.0	86.1	53.9	37.7	0.0	13.9	11.2	3.3	0.0	0.0	35.0	59.0
High pension income ( $\lambda = 1$ )	99.8	94.5	81.6	63.0	0.1	5.4	15.0	6.0	0.0	0.0	3.2	30.8

**Table III**  
**Welfare Analysis: Equivalent Increase in Financial Wealth (Percentage Points)**  
**of Having Access to Annuity Markets**

Table III reports welfare gains in the presence of annuity markets for all cases considered previously. The first, stylized case assumes a female with maximum life-span age 20 - 100, no initial endowment, no administration costs for annuities, no mortality asymmetries,  $RRA = 5$ ,  $EIS = 1/5$ , and a zero-bequest motive ( $k = 0$ ). The second case introduces administration costs for annuities  $\delta = 0.073$  and mortality asymmetries (2000 Population Basic vs. 1996 US Annuity 2000 Basic mortality table). The third case additionally considers a bequest motive ( $k = 2$ ). The remaining cases are variations of the third case with costs and bequest. Welfare gains are computed as the equivalent percentage increase in financial wealth an individual without access to annuity markets would need in order to attain the same expected utility as in the case with annuity markets. The computations are done for age 60, 70, 80, and 90.

<i>Case</i>	<i>Age</i>			
	60	70	80	90
Stylized case	14.41	16.00	23.75	49.83
With costs	9.54	12.79	16.51	31.16
With bequest	5.69	8.43	14.14	30.07
Males	5.35	8.95	18.75	41.31
Bad health	0.96	2.62	6.73	21.74
Low IES ( $\psi = 0.1$ )	0.40	1.18	3.68	14.70
High IES ( $\psi = 0.3$ )	8.34	11.87	21.30	43.80
Low RRA ( $\rho = 2$ )	0.00	0.00	0.42	0.10
Low pension income ( $\lambda = 0.5$ )	6.87	8.75	14.18	30.19
High pension income ( $\lambda = 1$ )	0.90	2.19	7.64	24.38